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The relativistic linear singular oscillator

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Abstract
Exactly solvable model of the linear singular oscillator in the relativistic configurational space is considered. We have found wavefunctions and energy spectrum for the model under study. It is shown that they have the correct non-relativistic limits.

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1. Introduction

In the framework of non-relativistic quantum mechanics [1–5] studied in detail the linear singular oscillator, which is described in the configuration representation by the Hamiltonian

$$H_N = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m\omega^2}{2}x^2 + \frac{g}{x^2}. \tag{1.1}$$

The singular oscillator, being one of the rare exactly solvable problems in non-relativistic quantum mechanics, has been extensively used in many applications. There are a lot of quantum mechanical and quantum field theory problems leading to the solution of the Schrödinger equation with Hamiltonian (1.1). For example, it served as an initial point in constructing exactly solvable models of interacting N-body systems [2]. It was also used for the modelling of diatomic [6] and polyatomic [7] molecules. The interest in the given Hamiltonian recently has increased in connection with its use for the description of spin chains [8], quantum Hall effect [9], fractional statistics and anyons [10].

In the case of a constant frequency ω, a complete set of orthonormalized eigenfunctions of Hamiltonian (1.1) in the interval $0 < x < \infty$ can be chosen in the form (see, for example, [4])

$$\psi_n^{\text{non-rel}}(x) = \left[2\left(\frac{m\omega}{\hbar}\right)^{d+1}\frac{n!}{\Gamma(d+n+1)}\right]^{\frac{1}{2}} x^{d+\frac{1}{2}}\,\mathrm{e}^{-\frac{m\omega}{2\hbar}x^2} L_n^d\left(\frac{m\omega}{\hbar}x^2\right) \tag{1.2}$$

where $d = \frac{1}{2}\sqrt{1+\frac{8mg}{\hbar^2}}$, and $L_n^d(y)$ are the associated Laguerre polynomials. The corresponding eigenvalues of H_N (1.1) have the form

$$E_n^{\text{non-rel}} = \hbar\omega(2n + d + 1) \qquad n = 0, 1, 2, 3, \ldots. \tag{1.3}$$

The purpose of this paper is to construct and investigate a relativistic exactly solvable model of the quantum linear singular oscillator (1.1), which can be applied for studying relativistic physical systems as well as systems on a lattice.

Our construction is based on the version of the relativistic quantum mechanics, which was developed in several papers and applied to the solution of a lot of problems in particle physics [11–24]. Although this version of the relativistic quantum mechanics is closely analogous to the non-relativistic quantum mechanics, its essential characteristic is that the relative motion wavefunction satisfies a finite-difference equation with a step equal to the Compton wavelength of the particle, $\lambda = \hbar/mc$. For example, in the case of a local quasipotential of interaction $V(\vec{r})$ the equation for the wavefunction of two scalar particles with equal mass has the form

$$[H_0 + V(\vec{r})]\psi(\vec{r}) = E\psi(\vec{r}) \tag{1.4}$$

where the finite-difference operator H_0 is a relativistic free Hamiltonian

$$H_0 = mc^2\left[\cosh(\mathrm{i}\lambda\partial_r) + \frac{\mathrm{i}\lambda}{r}\sinh(\mathrm{i}\lambda\partial_r) + \frac{\vec{L}^2}{2(mcr)^2}\exp(\mathrm{i}\lambda\partial_r)\right] \tag{1.5}$$

and $\vec{L}^2$ is the square of the angular momentum operator and $\partial_r \equiv \frac{\partial}{\partial r}$. The technique of difference differentiation was developed and analogues of the important functions of the continuous analysis were obtained to fit the relativistic quantum mechanics, based on equation (1.4) [12, 13].

Unlike non-relativistic quantum mechanics, because of the presence of the finite-difference operator $\mathrm{e}^{\mathrm{i}\lambda\partial_r}$, in the presented relativistic quantum mechanics both the Hamiltonian and the quasipotential contain imaginary terms. However, they are Hermitian operators with respect to scalar product $\int \psi^*\varphi\, \mathrm{d}\vec{r}$, where $\psi(\vec{r})$ and $\varphi(\vec{r})$ are the square-integrable functions.

We note that there is a regular method for construction of the quasipotential in the framework of the field-theoretical formalism. However, it can also be introduced phenomenologically.

The space of vectors $\vec{r}$ is called the relativistic configurational space or $\vec{r}$-space. The concept of $\vec{r}$-space has been introduced for the first time in the context of the quasipotential approach to the relativistic two-body problem [11].

The quasipotential equations for the relativistic scattering amplitude and the wavefunction $\psi(\vec{p})$ in the momentum space have the form [15, 25]

$$A(\vec{p},\vec{q}) = \frac{m}{4\pi}V(\vec{p},\vec{q};E_q) + \frac{1}{(2\pi)^3}\int V(\vec{p},\vec{k};E_q)G_q(k)A(\vec{k},\vec{q})\,\mathrm{d}\Omega_k \tag{1.6}$$

$$\psi(\vec{p}) = (2\pi)^3\delta(\vec{p}(-)\vec{q}) + \frac{1}{(2\pi)^3}G_q(p)\int V(\vec{p},\vec{k};E_q)\psi(\vec{k})\,\mathrm{d}\Omega_k \tag{1.7}$$

where

$$\begin{aligned} G_q(p) &= \frac{1}{E_q - E_p + \mathrm{i}0} \qquad & \delta(\vec{p}(-)\vec{q}) &\equiv \sqrt{1 + \frac{\vec{q}^2}{m^2c^2}}\delta(\vec{p} - \vec{q}) \\ \mathrm{d}\Omega_k &= \frac{\mathrm{d}\vec{k}}{\sqrt{1 + \frac{\vec{k}^2}{m^2c^2}}} \qquad & E_q &= \sqrt{\vec{q}^2c^2 + m^2c^4} \end{aligned} \tag{1.8}$$

and $V(\vec{p},\vec{k};E_q)$ is the quasipotential.

The integration in (1.6) and (1.7) is carried out over the mass shell of the particle with mass m, i.e. over the upper sheet of the hyperboloid ${p_0}^2 - \vec{p}^2 = m^2c^2$, which from the geometrical point of view realizes the three-dimensional Lobachevsky space. The group of motions of this space is the Lorentz group $SO(3, 1)$.

Equations (1.6) and (1.7) have absolute character with respect to the geometry of the momentum space, i.e., formally they do not differ from the non-relativistic Lippmann–Schwinger and Schrödinger equations. We can derive equations (1.6) and (1.7) substituting the relativistic (non-Euclidean) expressions for the energy, volume element and δ-function by their non-relativistic (Euclidean) analogues:

$$\begin{aligned} &E_q = \frac{q^2}{2m} \to E_q = \sqrt{\vec{q}^2c^2 + m^2c^4} \\ &\mathrm{d}\vec{k} \to \mathrm{d}\Omega_k = \frac{\mathrm{d}\vec{k}}{\sqrt{1 + \frac{\vec{k}^2}{m^2c^2}}} \\ &\delta(\vec{p} - \vec{q}) \to \delta(\vec{p}(-)\vec{q}). \end{aligned} \tag{1.9}$$

As a consequence of this geometrical treatment, the application of the Fourier transformation to the Lorentz group becomes natural instead of the usual one. In this case the relativistic configurational $\vec{r}$-space conception arises.

Transition to relativistic configurational $\vec{r}$-representation

$$\psi(\vec{r}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int \xi(\vec{p}, \vec{r})\psi(\vec{p})\,\mathrm{d}\Omega_p \tag{1.10}$$

is performed by the use of expansion on the matrix elements of the principal series of the unitary irreducible representations of the Lorentz group:

$$\begin{aligned} &\xi(\vec{p}, \vec{r}) = \left(\frac{p_0 - \vec{p}\vec{n}}{mc}\right)^{-1-\mathrm{i}r/\lambda} \\ &\vec{r} = r\vec{n} \qquad 0 \leqslant r < \infty \\ &\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \qquad p_0 = \sqrt{\vec{p}^2 + m^2c^2}. \end{aligned} \tag{1.11}$$

The quantity r is relativistic invariant and is connected with the eigenvalues of the Casimir operator $\hat{C} = \vec{N}^2 - \vec{L}^2$ in the following way:

$$C = \lambda^2 + r^2 \tag{1.12}$$

where $\vec{L}$ and $\vec{N}$ are the rotation and boost generators.

It is easy to verify that the function (the relativistic 'plane wave') (1.11) obeys the finite-difference Schrödinger equation

$$(H_0 - E_p)\xi(\vec{p}, \vec{r}) = 0. \tag{1.13}$$

The relativistic plane waves form a complete and orthogonal system of functions in the momentum Lobachevsky space [26].

If we perform the relativistic Fourier transformation (1.10) in equation (1.7), we arrive at the finite-difference Schrödinger equation (1.4) with the local (in the general case, non-local) potential in the relativistic $\vec{r}$-space.

In the relativistic $\vec{r}$-space the Euclidean geometry is realized and, in particular, there exists a momentum operator in the relativistic configurational $\vec{r}$ representation [13]

$$\hat{\vec{p}} = -\vec{n}(\mathrm{e}^{\mathrm{i}\lambda\partial_r} - H_0) - \vec{m}\frac{1}{r}\,\mathrm{e}^{\mathrm{i}\lambda\partial_r} \tag{1.14}$$

where a three-dimensional vector $\vec{m}$ has the following components [23]:

$$
\begin{aligned}
m_1 &= \mathrm{i}\left(\cos\varphi\cos\theta\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
m_2 &= \mathrm{i}\left(\sin\varphi\cos\theta\frac{\partial}{\partial\theta} - \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
m_3 &= -\mathrm{i}\sin\theta\frac{\partial}{\partial\theta}.
\end{aligned}
\tag{1.15}
$$

The components of (1.14) and free Hamiltonian obey the following commutation relations:

$$[\hat{p}_i, \hat{p}_j] = [\hat{p}_i, H_0] = 0 \qquad i, j = 1, 2, 3.$$

The relativistic plane wave is the eigenfunction of the operator $\hat{\vec{p}}$:

$$\hat{\vec{p}}\xi(\vec{p}, \vec{r}) = \vec{p}\xi(\vec{p}, \vec{r}). \tag{1.16}$$

This means that (1.11) describes the free relativistic motion with definite energy and momentum.

In the non-relativistic limit we come to the usual three-dimensional configurational space and relativistic plane wave (1.11) goes over into the Euclidean plane wave, i.e.

$$\lim_{c\to\infty}\xi(\vec{p}, \vec{r}) = \mathrm{e}^{\mathrm{i}\vec{p}\vec{r}/\hbar}. \tag{1.17}$$

Note that all the important exactly solvable cases of non-relativistic quantum mechanics (potential well, Coulomb potential, harmonic oscillator etc) are also exactly solvable for the case of equation (1.4).

2. Relativistic quantum mechanics: the one-dimensional case

In the one-dimensional case the relativistic plane wave takes the form [18]

$$\xi(p, x) = \left(\frac{p_0 - p}{mc}\right)^{-\mathrm{i}x/\lambda} = \left(\frac{p_0 + p}{mc}\right)^{\mathrm{i}x/\lambda} \tag{2.1}$$

or, in hyperpolar coordinates

$$p_0 = mc\cosh\chi \qquad p = mc\sinh\chi \tag{2.2}$$

we have

$$\xi(p, x) = \mathrm{e}^{\mathrm{i}x\chi/\lambda} \tag{2.3}$$

where $\chi = \ln\left(\frac{p_0+p}{mc}\right)$ is rapidity.

The one-dimensional plane waves obey the completeness and orthogonality conditions

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\xi(p, x)\xi^*(p, x')\,\mathrm{d}\Omega_p = \delta(x - x') \tag{2.4a}$$

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\xi(p, x)\xi^*(p', x)\,\mathrm{d}x = \delta(p(-)p') = \delta(mc(\chi - \chi')) \tag{2.4b}$$

where $\mathrm{d}\Omega_p = mc\frac{\mathrm{d}p}{p_0} = mc\,\mathrm{d}\chi$ is the invariant volume element in the one-dimensional Lobachevsky space, realized on the upper sheet of the hyperbola $p_0^2 - p^2 = m^2c^2$, $p_0 > 0$.

The free Hamiltonian and momentum operators are finite-difference operators

$$\hat{H}_0 = mc^2\cosh(\mathrm{i}\lambda\partial_x) \qquad \hat{p} = -mc\sinh(\mathrm{i}\lambda\partial_x). \tag{2.5}$$

Plane wave (2.1) obeys the free relativistic finite-difference Schrödinger equation

$$(\hat{H}_0 - E_p)\,\xi(p,x) = 0 \qquad E_p = cp_0 = c\sqrt{p^2 + m^2c^2} \tag{2.6}$$

and

$$\hat{p}\xi(p,x) = p\xi(p,x).$$

3. The finite-difference relativistic model of the linear singular oscillator

We consider a model of the relativistic linear singular oscillator, which corresponds to the following interaction potential:

$$V(x) = \left[\frac{1}{2}m\omega^2 x(x+\mathrm{i}\lambda) + \frac{g}{x(x+\mathrm{i}\lambda)}\right]\mathrm{e}^{\mathrm{i}\lambda\partial_x} \tag{3.1}$$

where g is a real quantity (we will assign a restriction for the values of the parameter g below).

For $g = 0$ it coincides with the quasipotential of the relativistic linear oscillator studied in detail in [18].

Let us note that in contrast to the case of the Coulomb potential [14, 23, 24], which can be calculated as an input of the one-photon exchange, the relativistic generalization of the oscillator or singular oscillator potential is not uniquely defined. Therefore, for construction of the quasipotential (3.1) we proceed from the following requirements for the quasipotential: (a) exact solubility; (b) the correct non-relativistic limit; (c) existence of the dynamical symmetry.

The operator (3.1) is Hermitian with respect to a scalar product

$$(\psi, \varphi) = \int_{-\infty}^{\infty} \psi^*(x)\varphi(x)\,\mathrm{d}x \tag{3.2}$$

i.e. $(V\psi, \varphi) = (\psi, V\varphi)$. Here functions $\psi(x)$ and $\varphi(x)$ vanish at $x = \pm\infty$ together with all their derivatives. In this connection, we note that the Hermitian conjugate of the operator $A = f(x)\,\mathrm{e}^{\gamma\partial_x}$ with respect to the scalar product (3.2) has the following form:

$$A^+ = \mathrm{e}^{-\gamma^*\partial_x} f^*(x)$$

where $f(x)$ is some complex function.

It is to be emphasized that potential (3.1) possesses the correct non-relativistic limit, i.e.

$$\lim_{c\to\infty} V(x) = \frac{1}{2}m\omega^2x^2 + \frac{g}{x^2}.$$

A relativistic singular oscillator is described by the following finite-difference equation:

$$\left[mc^2\cosh \mathrm{i}\lambda\partial_x + \frac{1}{2}m\omega^2 x(x+\mathrm{i}\lambda)\,\mathrm{e}^{\mathrm{i}\lambda\partial_x} + \frac{g}{x(x+\mathrm{i}\lambda)}\,\mathrm{e}^{\mathrm{i}\lambda\partial_x}\right]\psi(x) = E\psi(x) \tag{3.3}$$

with the boundary conditions for the wavefunction $\psi(0) = 0$ and $\psi(\infty) = 0$.

We shall confine ourselves to the interval $0 \leqslant x < \infty$.

In terms of dimensionless variable $\rho = \frac{x}{\lambda}$ and parameters $\omega_0 = \frac{\hbar\omega}{mc^2}$, $g_0 = \frac{mg}{\hbar^2}$ equation (3.3) takes the form

$$\left[\cosh \mathrm{i}\partial_\rho + \frac{1}{2}\omega_0^2\rho^{(2)}\,\mathrm{e}^{\mathrm{i}\partial_\rho} + \frac{g_0}{\rho^{(2)}}\,\mathrm{e}^{\mathrm{i}\partial_\rho}\right]\psi(\rho) = \frac{E}{mc^2}\psi(\rho) \tag{3.4}$$

where $\rho^{(2)} = \rho(\rho + \mathrm{i})$.

To solve equation (3.4) we choose $\psi(\rho)$ as

$$\psi(\rho) = c(-\rho)^{(\alpha)}\omega_0^{\mathrm{i}\rho}\Gamma(\nu+\mathrm{i}\rho)\Omega(\rho) \equiv c(-\rho)^{(\alpha)}M(\rho)\Omega(\rho) \tag{3.5}$$

where α and ν are arbitrary constant parameters, which will be defined below.

Functions

$$(-\rho)^{(\alpha)} = \mathrm{i}^{\alpha}\frac{\Gamma(\mathrm{i}\rho+\alpha)}{\Gamma(\mathrm{i}\rho)} \qquad \text{and} \qquad M(\rho) = \omega_0^{\mathrm{i}\rho}\Gamma(\nu+\mathrm{i}\rho) \tag{3.6}$$

are connected with the behaviour of the wavefunction $\psi(\rho)$ at points $\rho = 0$ and $\rho = \infty$, respectively.

Inserting (3.5) into (3.4), we obtain

$$\left[(\alpha+\mathrm{i}\rho)(\nu+\mathrm{i}\rho)\,\mathrm{e}^{-\mathrm{i}\partial_\rho} - \frac{\omega_0^2(\rho^{(2)})^2+\rho^{(2)}+2g_0}{\omega_0^2(\alpha-1+\mathrm{i}\rho)(\nu-1+\mathrm{i}\rho)}\,\mathrm{e}^{\mathrm{i}\partial_\rho}\right]\Omega(\rho) = 2\mathrm{i}\epsilon\rho\Omega(\rho) \tag{3.7}$$

where $\epsilon = E/\hbar\omega$ is the dimensionless energy.

Now we choose constant parameters α and ν in such a way that they satisfy the following relation:

$$\frac{\omega_0^2(\rho^{(2)})^2+\rho^{(2)}+2g_0}{\omega_0^2(\alpha-1+\mathrm{i}\rho)(\nu-1+\mathrm{i}\rho)} = (\alpha-\mathrm{i}\rho)(\nu-\mathrm{i}\rho). \tag{3.8}$$

Hence we get

$$\alpha = \frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2}{\omega_0^2}\big(1-\sqrt{1-8g_0\omega_0^2}\big)} \tag{3.9a}$$

$$\nu = \frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{2}{\omega_0^2}\big(1+\sqrt{1-8g_0\omega_0^2}\big)}. \tag{3.9b}$$

Then the function $\Omega(\rho)$ will satisfy the difference equation

$$[(\alpha+\mathrm{i}\rho)(\nu+\mathrm{i}\rho)\,\mathrm{e}^{-\mathrm{i}\partial_\rho} - (\alpha-\mathrm{i}\rho)(\nu-\mathrm{i}\rho)\,\mathrm{e}^{\mathrm{i}\partial_\rho}]\Omega(\rho) = 2\mathrm{i}\epsilon\rho\Omega(\rho). \tag{3.10}$$

By substitution of the $\Omega(\rho)$ function expansion as the following power series

$$\Omega(\rho) = \sum_{k=0}^{\infty} e_k(\mathrm{i}\rho)^k \tag{3.11}$$

into (3.10) one finds that the coefficients at odd degrees of $\mathrm{i}\rho$ become zero (i.e., all $e_{2k+1} = 0, k = 0, 1, 2, 3, \ldots$), but the coefficients at even degrees of $\mathrm{i}\rho$ satisfy the following recurrence relation:

$$(\epsilon-\alpha-\nu-2j)e_{2j} = \sum_{k=j+1}^{\infty}\left[\alpha\nu C_{2k}^{2j+1} + (\alpha+\nu)C_{2k}^{2j} + C_{2k}^{2j-1}\right]e_{2k} \tag{3.12}$$

where C_n^m are binomial coefficients. From (3.12) it follows that the power series (3.11) will be terminated at the term $e_{2n}(\mathrm{i}\rho)^{2n}$ if condition $\epsilon \equiv \epsilon_n = 2n+\alpha+\nu, n = 0, 1, 2, 3, \ldots$ holds. This gives the following quantization rule for the energy spectrum for relativistic singular oscillator (3.1):

$$E_n = \hbar\omega\epsilon_n = \hbar\omega(2n+\alpha+\nu) \qquad n = 0, 1, 2, 3, \ldots. \tag{3.13}$$

Hence in the case of (3.13) solutions of equation (3.10) coincide with the continuous dual Hahn polynomials

$$\Omega(\rho) \equiv \Omega_n(\rho) = S_n\left(\rho^2; \alpha, \nu, \tfrac{1}{2}\right) \tag{3.14}$$

defined with relation [27]

$$S_n(x^2; a, b, c) = (a+b)_n (a+c)_n \, {}_3F_2 \left(\begin{matrix} -n & a+\mathrm{i}x & a-\mathrm{i}x \\ a+b & a+c & \end{matrix} \middle| \, 1 \right) \tag{3.15}$$

where $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

Continuous dual Hahn polynomials (3.15) satisfy the three-term recurrence relation [27]

$$(A_n + C_n - a^2 - x^2)\, \tilde{S}_n(x^2) = A_n \tilde{S}_{n+1}(x^2) + C_n \tilde{S}_{n-1}(x^2) \tag{3.16}$$

where $A_n = (n+a+b)(n+a+c)$, $C_n = n(n+b+c-1)$ and

$$\tilde{S}_n(x^2) = \frac{S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n}.$$

Hence normalized wavefunctions for the stationary states of the relativistic singular linear oscillator have the following form:

$$\begin{aligned} \psi_n(\rho) &= c_n (-\rho)^{(\alpha)} \omega_0^{\mathrm{i}\rho} \Gamma(\nu + \mathrm{i}\rho)\, S_n\big(\rho^2; \alpha, \nu, \tfrac{1}{2}\big) \\ c_n &= \sqrt{\frac{2}{\Gamma(n+\alpha+\nu)\Gamma\big(n+\alpha+\frac{1}{2}\big)\Gamma\big(n+\nu+\frac{1}{2}\big) n!}}. \end{aligned} \tag{3.17}$$

Wavefunctions (3.17) are orthonormalized as follows:

$$\int_0^\infty \psi_n(\rho)\psi_m^*(\rho)\,\mathrm{d}\rho = \delta_{nm}.$$

One can verify that the difference Hamiltonian of equation (3.3)

$$H = mc^2[a^+a^- + \omega_0(\alpha+\nu)] \tag{3.18}$$

may be factorized in terms of the operators, having the form

$$a^- = \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\partial_\rho} - \omega_0\, \mathrm{e}^{\frac{\mathrm{i}}{2}\partial_\rho} (\nu + \mathrm{i}\rho) \left(1 + \frac{\alpha}{\mathrm{i}\rho}\right) \right] \tag{3.19a}$$

$$a^+ = \frac{1}{\sqrt{2}}\left[\mathrm{e}^{-\frac{\mathrm{i}}{2}\partial_\rho} - \omega_0 (\nu - \mathrm{i}\rho) \left(1 - \frac{\alpha}{\mathrm{i}\rho}\right) \mathrm{e}^{\frac{\mathrm{i}}{2}\partial_\rho} \right]. \tag{3.19b}$$

They are the pair of Hermitian conjugate operators. Using a^- and a^+, one can construct a dynamical symmetry algebra of the relativistic linear singular oscillator (3.3).

We note that the Hermiticity condition of Hamiltonian H imposes a restriction on the values of the quantity g. Indeed, from expression (3.18) it follows that H is a Hermitian operator and its eigenvalues E_n (3.13) are real only in the case when α and ν are real or complex-conjugate quantities. Therefore, g must satisfy the condition $g > -\frac{\hbar^2}{8m} - \frac{\hbar^2\omega_0^2}{32m}$. The case $g < -\frac{\hbar^2}{8m} - \frac{\hbar^2\omega_0^2}{32m}$ ('collapse') will be considered separately.

Since the parameter $\mu = \frac{mc^2}{\hbar\omega} \to \infty$ when $c \to \infty$, by using (A.1) in the appendix it is easy to show that in the limit case, when the velocity of the light approaches ∞, wavefunctions (3.17) coincide with wavefunctions (1.2) of the non-relativistic linear singular oscillator.

Energy spectrum (3.13) also has a correct non-relativistic limit, i.e.

$$E_n - mc^2 \to E_n^{\text{non-rel}} = \hbar\omega(2n + d + 1).$$

By the use of (A.2) it can be shown that, in the $g \to 0$ limit, the eigenfunctions (3.17) and the eigenvalues (3.13) transform into the 'antisymmetrical' eigenfunctions and eigenvalues of

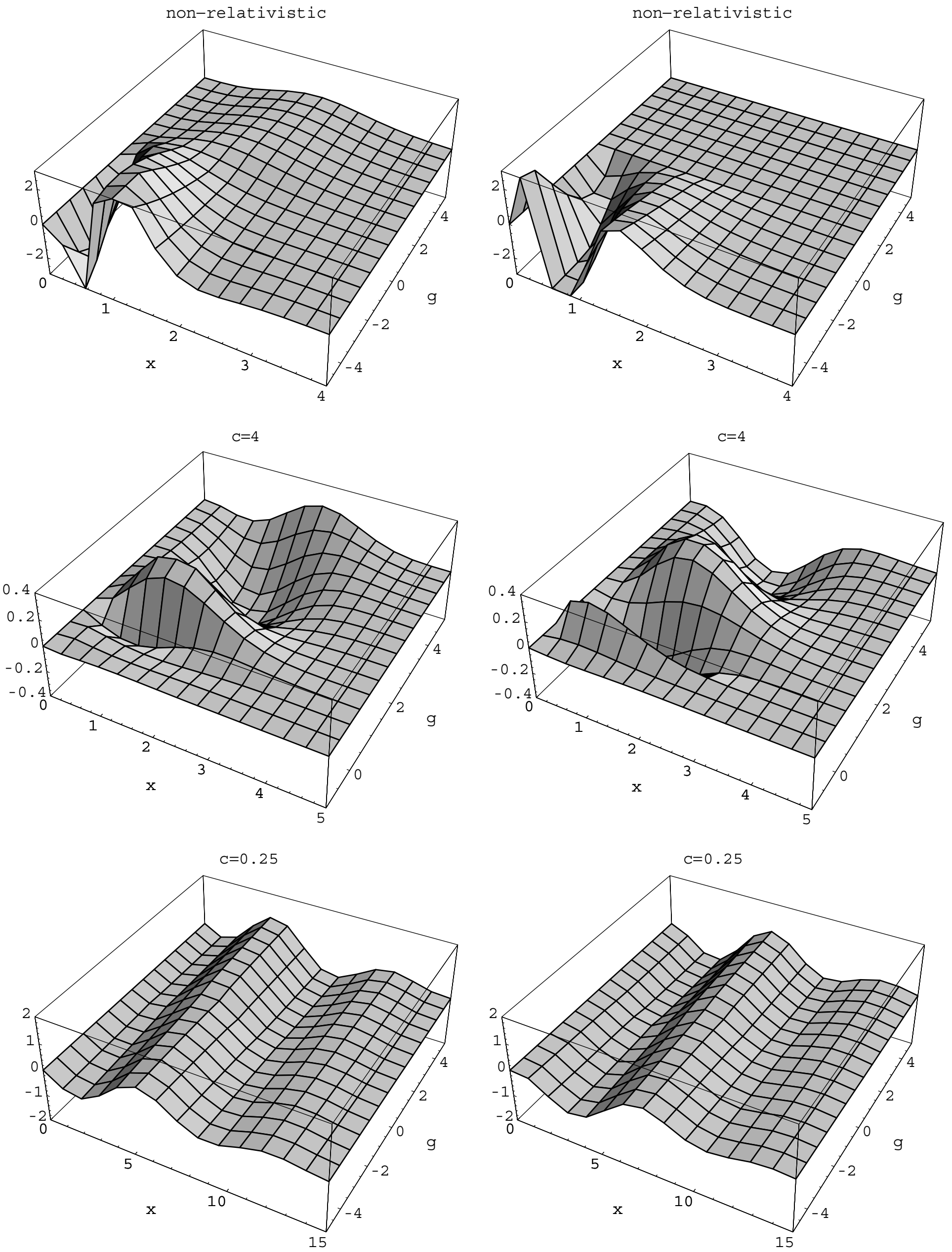

Figure 1. The behaviour of the ground state wavefunction of the relativistic linear singular oscillator (3.17) for values of the speed of light $c = \infty$ (the non-relativistic case), 4 and 0.25 ($m = \omega = \hbar = 1$). Real parts are shown in the left-hand plots, imaginary parts in the right-hand plots.

the relativistic linear harmonic oscillator, considered in [18]:

$$\psi_n^{\text{relosc}}(\rho) = c_n'[\nu'(\nu'-1)]^{-\mathrm{i}\rho/2}\Gamma(\rho+\mathrm{i}\nu')P_{2n+1}^{\nu'}\left(\rho;\frac{\pi}{2}\right)$$

$$E_n^{\text{relosc}} = \hbar\omega(2n+1+\nu') \qquad n = 0, 1, 2, 3, \ldots.$$

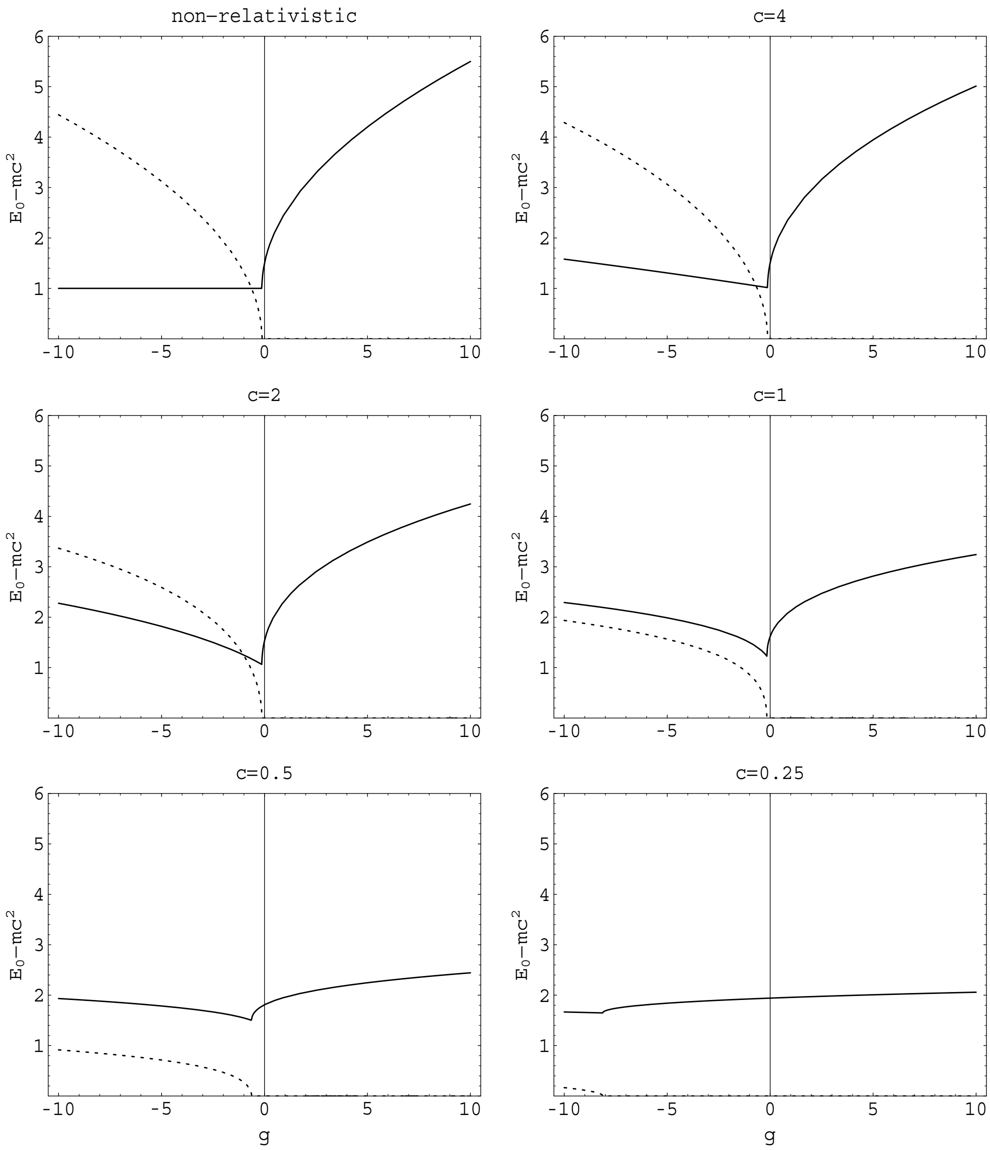

Figure 2. The behaviour of the ground state energy-level of the relativistic linear singular oscillator (3.13) for values of the speed of light $c = \infty$ (the non-relativistic case), 4, 2, 1, 0.5 and 0.25 ($m = \omega = \hbar = 1$). Real parts are shown by solid lines, imaginary parts by dashed lines. We see that decreasing c changes the appearance point of the imaginary part ('collapse' point $g < -\frac{1}{8}(1 + \frac{1}{4c^4})$), too.

Here $P_n^{\nu'}(\rho; \varphi)$ are the Meixner–Pollaczek polynomials and

$$c'_n = 2^{\nu'} \sqrt{\frac{(2n+1)!}{2\pi\lambda\Gamma(2n+1+\nu')}} \qquad \nu' = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\omega_0^2}}.$$

In figure 1 we show the behaviour of the ground state wavefunction $\psi_0(x)$ (3.17) depending on x and g for various values of the speed of light c. We see that except for the non-relativistic case, where $\psi_0(x)$ is real for $g \geqslant -\frac{\hbar^2}{8m}$, in the relativistic case the ground state wavefunction is complex and has a large number of zeros. Similarly, in figure 2 we show the behaviour of the ground state energy-level (3.13) depending on the g for various values of the speed of light c. From these plots we see that decreasing c changes the appearance point of the imaginary part ('collapse' point $g < -\frac{\hbar^2}{8m} - \frac{\hbar^2\omega_0^2}{32m}$).

4. Conclusion

The application of the finite-difference relativistic quantum mechanics to a large class of physical problems requires relativistic generalizations of the exactly solvable problems of non-relativistic quantum mechanics.

In this paper, we construct a relativistic model of the linear singular oscillator and explicitly solve the corresponding finite-difference equation. We determine eigenvalues and eigenfunctions of the problem. They have the correct non-relativistic limits. As in the non-relativistic case [2], the simplicity of the obtained energy spectrum suggests that a solution by group-theoretical methods should also be possible. Furthermore, we study some properties of the continuous dual Hahn polynomials.

We hope that the model of the relativistic linear oscillator proposed in this paper will be applied in future in various fields of quantum physics and likewise the non-relativistic singular oscillator.

Appendix

Here we present some formulae for the continuous dual Hahn polynomials (3.15) used in the text.

1. From the recurrence relation for the continuous dual Hahn polynomials (3.16), it can be shown that the following limit to the Laguerre polynomials holds:

$$\lim_{\mu\to\infty} \frac{1}{n!\mu^n} S_n\left(z\mu; a, b, \frac{1}{2}\right) = L_n^{a_0-\frac{1}{2}}(z) \tag{A.1}$$

where $a_0 = \lim\limits_{\mu\to\infty} a$ and $\lim\limits_{\mu\to\infty}(b-\mu) = \text{const}$.

2. It can be shown that in some particular cases continuous dual Hahn polynomials coincide with the Meixner–Pollaczek polynomials, i.e.

$$P_{2n+1}^b\left(x; \frac{\pi}{2}\right) = (-1)^n \frac{2^{2n+1}}{(2n+1)!} x S_n\left(x^2; 1, b, \frac{1}{2}\right) \tag{A.2}$$

and

$$P_{2n}^b\left(x; \frac{\pi}{2}\right) = (-1)^n \frac{2^{2n}}{(2n)!} S_n\left(x^2; 1, b, \frac{1}{2}\right). \tag{A.3}$$

To prove formulae (A.2) and (A.3) let us compare equations [27]

$$[\mathrm{e}^{-\mathrm{i}\varphi}(b+\mathrm{i}x)\,\mathrm{e}^{-\mathrm{i}\partial_x} - \mathrm{e}^{\mathrm{i}\varphi}(b-\mathrm{i}x)\,\mathrm{e}^{\mathrm{i}\partial_x}]y_1(x) = 2\mathrm{i}[x\cos\varphi - (k+b)\sin\varphi]y_1(x) \tag{A.4}$$

and

$$[(a+\mathrm{i}x)(b+\mathrm{i}x)\,\mathrm{e}^{-\mathrm{i}\partial_x} - (a-\mathrm{i}x)(b-\mathrm{i}x)\,\mathrm{e}^{\mathrm{i}\partial_x}]y_2(x) = 2\mathrm{i}x(2n+a+b)y_2(x) \tag{A.5}$$

where $y_1(x) = P_k^b(x, \varphi)$ are the Meixner–Pollaczek polynomials and $y_2(x) = S_n\left(x^2; a, b, \frac{1}{2}\right)$ are the continuous dual Hahn polynomials. It is easy to verify that, when $\varphi = \frac{\pi}{2}$ and $k = 2n + 1$, equation (A.4) coincides with equation (A.5) for $a = 1$. This means

$$P_{2n+1}^{b}\left(x; \frac{\pi}{2}\right) = N_n x S_n \left(x^2; 1, b, \frac{1}{2}\right). \tag{A.6}$$

Comparing coefficients, for example, for x^{2n+1} on the left-hand and right-hand sides of (A.6) we find that

$$N_n = (-1)^n \frac{2^{2n+1}}{(2n+1)!}.$$

One can prove relation (A.3) in the same way.

3. In the non-relativistic limit, when $\mu = \frac{mc^2}{\hbar\omega} \to \infty$ we have

$$\begin{aligned} &\lim_{\mu\to\infty} \alpha = d + \tfrac{1}{2} \\ &\lim_{\mu\to\infty} (\nu - \mu) = \tfrac{1}{2} \\ &\lim_{\mu\to\infty} (-\rho)^{(\alpha)} = \mathrm{e}^{\frac{1}{2}(d+\frac{1}{2})\ln\mu}(-\xi)^{d+\frac{1}{2}} \\ &\lim_{\mu\to\infty} M(\rho) = \sqrt{2\pi}\,\mathrm{e}^{\mu\ln\mu-\mu-\frac{\xi^2}{2}} \\ &\lim_{\mu\to\infty} c_n = \frac{1}{\pi\sqrt{n!\Gamma(n+d+1)}}\,\mathrm{e}^{\mu-(\mu+n+\frac{d}{2})\ln\mu} \end{aligned} \tag{A.7}$$

where $\xi = \sqrt{\frac{m\omega}{\hbar}}x$. To obtain these formulae we used the representation

$$\Gamma(z) \simeq \sqrt{\frac{2\pi}{z}}\,\mathrm{e}^{z\ln z - z} \qquad |z| \to \infty$$

for the gamma function.

4. In the non-relativistic limit we have the following limit relations for operators (3.19*a*) and (3.19*b*)

$$\lim_{\mu\to\infty} \sqrt{\mu}a^- = c^- = -\frac{\mathrm{i}}{\sqrt{2}}\left(\partial_\xi + \xi - \frac{d+\frac{1}{2}}{\xi}\right)$$

$$\lim_{\mu\to\infty} \sqrt{\mu}a^+ = c^+ = \frac{\mathrm{i}}{\sqrt{2}}\left(-\partial_\xi + \xi - \frac{d+\frac{1}{2}}{\xi}\right).$$

Hamiltonian (3.18) in this limit coincides with the Hamiltonian of the non-relativistic singular oscillator (1.1):

$$\lim_{\mu\to\infty} H = H_N = \hbar\omega(c^+c^- + d + 1) = \frac{\hbar\omega}{2}\left(-\partial_\xi{}^2 + \xi^2 + \frac{d^2 - \frac{1}{4}}{\xi^2}\right).$$

References

[1] Landau L D and Lifshitz E M 1997 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Butterworth-Heinemann)

[2] Calogero F 1969 *J. Math. Phys.* **10** 2191
Calogero F 1971 *J. Math. Phys.* **12** 419

[3] Camiz P, Gerardi A, Marchioro, Presutti E and Scacciatelli E 1971 *J. Math. Phys.* **12** 2040

[4] Dodonov V V, Malkin I A and Manko V I 1974 *Physica* **72** 597

[5] Dodonov V V, Manko V I and Rosa L 1998 *Phys. Rev.* A **57** 2851

[6] Chumakov S M, Dodonov V V and Manko V I 1986 *J. Phys. A: Math. Gen.* **19** 3229
[7] Hartmann A 1972 *Theor. Chim. Acta* **24** 201
[8] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
[9] Frahm H 1993 *J. Phys. A: Math. Gen.* **69** L473
[10] Leinaas J M and Myrhein J 1988 *Phys. Rev.* B **37** 9286
[11] Kadyshevsky V G, Mir-Kasimov M R and Skachkov N B 1968 *Nuovo Cimento* **55** 233
[12] Kadyshevsky V G, Mir-Kasimov M R and Skachkov N B 1969 *Yad. Fiz.* **9** 212
[13] Kadyshevsky V G, Mir-Kasimov R M and Freeman M 1969 *Yad. Fiz.* **9** 646
[14] Freeman M, Mateev M D and Mir-Kasimov R M 1969 *Nucl. Phys.* B **12** 197
[15] Kadyshevsky V G, Mir-Kasimov R M and Skachkov N B 1972 *Phys. Elem. Part. At. Nucl.* **2** 635
[16] Klein A and Lee H T-S 1974 *Phys. Rev.* D **10** 4308
[17] Amirkhanov I V, Grusha G V and Mir-Kasimov R M 1981 *Phys. Elem. Part. At. Nucl.* **12** 651
[18] Atakishiyev N M, Mir-Kasimov R M and Nagiyev Sh M 1980 *Theor. Math. Phys.* **44** 47
[19] Atakishiyev N M, Mir-Kasimov R M and Nagiyev Sh M 1985 *Ann. Phys., Lpz* **42** 25
[20] Atakishiyev N M, Mir-Kasimov R M and Nagiyev Sh M 1986 *Theor. Math. Phys.* **67** 68
[21] Kagramanov E D, Mir-Kasimov R M and Nagiyev Sh M 1989 *Phys. Lett.* A **140** 1
[22] Kagramanov E D, Mir-Kasimov R M and Nagiyev Sh M 1990 *J. Math. Phys.* **31** 1733
[23] Nagiyev Sh M 1988 *J. Phys. A: Math. Gen.* **21** 2559
[24] Nagiyev Sh M 1989 *Theor. Math. Phys.* **80** 40
[25] Kadyshevsky V G 1968 *Nucl. Phys.* **6** 125
[26] Shapiro I S 1956 *Sov. Phys.–Dokl.* **1** 91
[27] Koekoek R and Swarttouw R F 1998 *Delft University of Technology Report* no 98-17